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Symbolic dynamics for strong chaos on stochastic webs: General quasisymmetry

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An exact symbolic dynamics is introduced for a model of strong chaos on stochastic webs with *arbitrary quasisymmetry* (*no* translational invariance, in general). The model describes the interaction of charged particles with an electrostatic wave packet in a magnetic field under arbitrary resonance or nonresonance conditions. As a first application, the symbolic dynamics is used to identify random chaotic sets and to calculate accurately their global diffusion rate as a function of the resonance parameter.

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Symbolic dynamics, i.e., the representation of classical orbits by infinite sequences of symbols, is a useful tool for understanding basic properties of chaos in dynamical systems, and for the computation of several quantities characterizing the chaotic motion [1]. A good symbolic dynamics, with well-defined (and relatively simple) alphabet and grammar, has been introduced and extensively used in several interesting cases of dissipative systems [2]. In the case of Hamiltonian systems, on the other hand, a satisfactory symbolic dynamics is usually more difficult to define, mainly due to the intricate mixture of chaotic and regular motions on all scales of phase space.

Some progress has been made recently [3] by determining an approximate generating partition for the standard map in a strongly chaotic regime where the stability islands cover a small area of phase space. Several rigorous results are known for piecewise linear area-preserving maps [4–8]. A simple but now classic example is the sawtooth map [5–12], which, like the standard map, belongs to the class of Taylor-Chirikov (TC) twist maps (kicked-rotor Hamiltonians). Thus, while this map is uniformly hyperbolic (completely chaotic), it features much of the unstable regular structure (ordered periodic orbits, cantori, and resonances) generically found in typical maps for large nonlinearity parameter [6–8]. As such, it turned out to be a good model for studying some basic aspects of Hamiltonian dynamics [5–8,11,12] as well as for checking the validity of transport theories [8–10] in the strong chaos limit. A symbolic dynamics for the sawtooth map, which is both exact and nontrivial [13], has been worked out in detail [6–8] and has been used in several applications [6–9,11,12]. In particular, it was shown [12] that this symbolic dynamics provides a simple but powerful method for investigating the quasiregularity of chaos in general TC maps for arbitrary values of the nonlinearity parameter.

In this paper, the exact symbolic dynamics of the sawtooth map will be considerably extended to a much richer dynamical model, representing a class of Hamiltonian systems with properties qualitatively different from those of TC maps: charged particles interacting with an electrostatic wave packet in a uniform magnetic field **B**. It is assumed that the wave packet propagates perpendicularly to **B**, a case of major interest in plasma physics [14]. If one also assumes that the wave packet has a very broad frequency spectrum, the time dependence can be approximated by a periodic δ function [15]. The general Hamiltonian for these systems is then [15,16]

$$H=\omega^2(u^2+v^2)/2+\bar{K}V(x_c-v)\sum_{s=-\infty}^{\infty}\delta(t-sT), \qquad (1)$$

where ω is the cyclotron frequency, (u,v) are conjugate variables giving the kinetic momentum in a magnetic field, $\bar{K}$ is a parameter, $V(x)$ is the periodic potential for the wave packet, x_c is the *conserved* coordinate of the cyclotron-orbit center, and T is the time period. The Poincaré map for (1) is the "web map" [15]

$$M:\quad \begin{cases} u_{s+1}=[u_s+Kg(v_s)]\cos\alpha+v_s\sin\alpha, \\ v_{s+1}=-[u_s+Kg(v_s)]\sin\alpha+v_s\cos\alpha, \end{cases} \qquad (2)$$

where (u_s,v_s) are the values of (u,v) at time $sT-0$, $K\equiv\bar{K}/\omega$, $g(v)\equiv f(x_c-v)$ $[f(x)=-dV(x)/dx]$ is the force function, and $\alpha=\omega T$. Numerical studies [15] of the map (2) indicate that unbounded chaotic diffusion in the (u,v) phase plane occurs for arbitrarily small values of K, provided the initial kinetic energy is large enough. This is in contrast with TC maps, where this diffusion generally takes place only for $K>K_c\neq 0$, when the last bounding invariant curve disappears. The existence of such curves for (1) cannot be established on the basis of Kolmogorov-Arnol'd-Moser (KAM) theory, since the harmonic-oscillator Hamiltonian $H_0=\omega^2(u^2+v^2)/2$ is degenerate (linear in the action), unlike the rotor Hamiltonian for TC maps. The unbounded diffusion for (2) takes place on a "stochastic web" [15–19] (see Fig. 1), whose symmetry and structure are determined by the parameters α, x_c, and K. This makes the dynamics of (2) much richer than that of TC maps. Rational values of $\alpha/2\pi=m/n$ (m and n are coprime integers) correspond to resonance conditions. For $n=3,4,6$, the web has crystalline symmetry (triangular, square, hexagonal), see Fig. 1(a), and the map M^n is translationally invariant in phase space, like TC maps. In general, however, the system is *not* translationally invariant but only "quasisymmetric" [15] [see the quasicrystalline case $\alpha/2\pi=1/5$ in Fig. 1(b)].

In our model the function $f(-v)$ [giving $g(v)=f(x_c-v)$ in (2)] will be chosen as a sawtooth, $f(-v)=v$ for $-0.5\leq v<0.5$ and $f(v+1)=f(v)$ [20]. This gives, for al-

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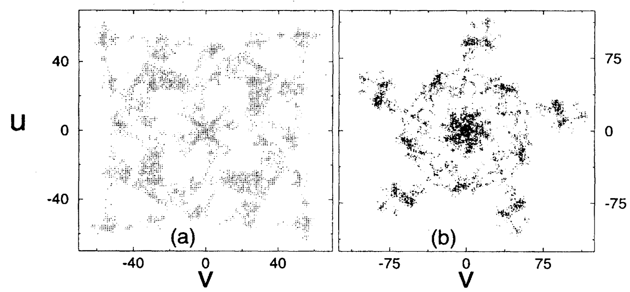

FIG. 1. Chaotic orbits forming transient stochastic webs for the model defined in the text: (a) $\alpha=2\pi/4$ (crystalline square), (b) $\alpha=2\pi/5$ (quasicrystalline pentagonal). In both cases $x_c=0$ and Lyapunov multiplier $\lambda=3.8$. These orbits have been generated by the symbolic dynamics introduced in the text (no roundoff errors), using in both cases a sequence **c** of length 20 000 (from random-number generator) and containing only two symbols, $c_s=-1,0$. Such a sequence is always admissible for $\lambda=3.8$ [see (12)].

most all values of K, a uniformly hyperbolic map (2) (see below). However, it usually takes a very long time for a chaotic orbit to fill ergodically a given region of phase space. During this time the motion exhibits the typical structure of stochastic webs (see Fig. 1) found for smooth $g(v)$ [15–19]. Thus, as in the case of the TC sawtooth map, our model should give a good description of typical maps (2) in the strong chaos limit. We show that this model has an exact [13] symbolic dynamics for *general quasisymmetry* (arbitrary α). As far as we are aware, this seems to be the first example of a symbolic dynamics describing a nontranslationally invariant system in an infinite phase space. As a first application, we identify random chaotic sets for the system and calculate accurately (without roundoff errors) their global diffusion rate as a function of α.

Our starting point is the "Newton" equation, which is easily derived from (2):

$$v_{s+1}-\eta v_s+v_{s-1}=\epsilon g(v_s), \quad (3)$$

where $\eta=2\cos\alpha$ and $\epsilon=-K\sin\alpha$. The variable v can be expressed in a unique way as $v=a+w+x_c$, where a is an integer and $-0.5\leq w<0.5$. In the sawtooth case, $g(v)=w$, and Eq. (3) can then be written as follows:

$$w_{s+1}-(\eta+\epsilon)w_s+w_{s-1}=-[b_s+(2-\eta)x_c], \quad (4)$$

where

$$b_s=a_{s+1}-\eta a_s+a_{s-1}. \quad (5)$$

It is easily verified that $\eta+\epsilon=\mathrm{Tr}(DM)=\lambda+\lambda^{-1}$, where $\lambda^{\pm 1}$ are the eigenvalues of the linearized map DM. For $|\eta+\epsilon|>2$, λ is real and we can assume that $|\lambda|>1$ ($\sigma=\ln|\lambda|$ is the Lyapunov exponent). In this uniformly hyperbolic case, Eq. (4) can be inverted by Green function methods (in analogy to the TC sawtooth map [6]) to express w_s in terms of the sequence $\{b_s\}$:

$$w_s=\frac{1}{\lambda-\lambda^{-1}}\sum_{s'=-\infty}^{\infty}\lambda^{-|s'|}[b_{s'+s}+(2-\eta)x_c]. \quad (6)$$

Given the integer sequence $\mathbf{a}=\{a_s\}$, the sequence $\mathbf{b}=\{b_s\}$ can be calculated from Eq. (5), and w_s is then fixed by Eq. (6). The orbit $v_s=a_s+w_s+x_c$ is thus uniquely determined from the sequence **a**, and vice versa, of course. One can associate with some orbit point, say v_0, a sequence $\mathbf{a}=\{a_{s'}\}$, where $a_{s'}$ is the value of a for the s'th iterate of v_0. The orbit point v_s will then obviously correspond to the "shifted" sequence $\mathbf{a}^{(s)}=\{a_{s'}^{(s)}=a_{s'+s}\}$. Thus, the encoding of orbits by sequences **a** provides a symbolic dynamics. The generating partition consists of the infinite strips $-0.5\leq v-x_c-a<0.5$, for all integers a. A sequence **a** is admissible only if the corresponding sequence **b** in (6) satisfies the "pruning" rule

$$-0.5\leq w_s<0.5 \quad (7)$$

for all s. As it stands, however, a_s is not a useful code since it is generally unbounded, and the alphabet is then not finite. A remedy for this problem will now be proposed.

Using (7) in (4), we find that b_s is bounded, $|b_s+(2-\eta)x_c|\leq 1+|\eta+\epsilon|/2$. Now, b_s can be expressed in a unique way as

$$b_s=c_s+r_s, \quad (8)$$

where $c_s=[b_s]$ is the integer part of b_s, and $r_s=b_s \bmod 1$ is the "remainder." After substituting (8) in (5), we obtain

$$a_{s+1}=c_s-[-\eta a_s]-a_{s-1}, \quad (9)$$

$$r_s=(-\eta a_s) \bmod 1. \quad (10)$$

Using Eqs. (8) and (10) in $|b_s+(2-\eta)x_c|\leq 1+|\eta+\epsilon|/2$, we get the inequality

$$-1-|\eta+\epsilon|/2-\beta<c_s+(2-\eta)x_c<1+|\eta+\epsilon|/2. \quad (11)$$

Here $\beta\equiv\sup\{r_s\}=0$ in the crystalline cases ($\eta=0,\pm 1$), $\beta=1$ for irrational η, and $\beta=(q-1)/q$ for rational $\eta=l/q$ (l and q are coprime integers). Relations (9)–(11) are our main results. Given an arbitrary "initial pair" (a_{-1},a_0) [fixed by the initial conditions (u_0,v_0)], the Diophantine equation (9) allows one to determine the basic sequence **a** from the sequence **c** of bounded integers c_s in the interval (11). Thus, the use of the sequence **c** effectively reduces the infinite alphabet of a_s to a finite one. For example, for $\lambda=3.5$, $x_c=0$, and irrational η ($\beta=1$), the effective alphabet consists of six symbols, $c_s=-3,\ldots,2$. We denote by Σ_c the space of admissible sequences **c** [satisfying (7)] for a given initial pair (a_{-1},a_0). In the crystalline cases, the map M^n ($n=3,4,6$) is translationally invariant and its dynamics can be reduced to a torus (the unit cell). Then, clearly, Σ_c does not depend on (a_{-1},a_0), and $c_s=b_s$ becomes a natural code for describing the toroidal dynamics using a finite generating partition, as in the case of TC maps [6]. In the absence of translational invariance, however, Σ_c generally depends on (a_{-1},a_0), and c_s should be considered only as an "auxiliary code" to the main code a_s. Despite this, one should notice that spaces Σ_c associated with different pairs (a_{-1},a_0) will generally correspond to orbit spaces having large intersections, since almost all pairs (a,a') will eventually be "reached" by most of the orbits (i.e., $a_s=a$

and $a_{s+1}=a'$ for some s); due to ergodicity. Moreover, some subspaces of Σ_c do not depend on (a_{-1},a_0), as shown below.

An interesting case is that of rational $\eta=l/q$. For sufficiently strong chaos ($|\lambda|\gg1$), one can always find admissible sequences **c** and $\bar{\mathbf{c}}$ related by $c_s=q\bar{c}_s+q\bar{r}_s+2j+[-\eta j]$, where $\bar{r}_s$ are the remainders corresponding to $\bar{c}_s$ ($q\bar{r}_s$ are integers), and j is some integer in $[0,q)$. Using Eq. (9), it is then easy to show that there exist sequences **a** and $\bar{\mathbf{a}}$, corresponding to **c** and $\bar{\mathbf{c}}$, which are related by $a_s=q\bar{a}_s+j$. This "self-similarity" implies that the orbit associated with **a** visits only a very regular subset of the generating partition, i.e., a periodic array of strips separated from each other by a constant distance q.

For periodic orbits (POs) with minimal period p ($v_{s+p}=v_s$), the sequences **a** and **b** are also periodic with minimal period p. However, in the absence of translational invariance (η not integer), the minimal period of **c** is generally only a divisor of p. This is because the periodicity of **b** implies that of **c**, but the contrary is generally not true if η is not integer, as one can see from Eqs. (8)–(10). Thus, in general, all the POs with period p and initial pair (a_{-1},a_0) can be systematically found as follows: (a) For each sequence **c** with alphabet in (11) and minimal period p' dividing p, check whether $(a_{p-1},a_p)=(a_{-1},a_0)$, where a_s is determined from Eq. (9). (b) If $(a_{p-1},a_p)=(a_{-1},a_0)$, check whether **c** is admissible, i.e., it satisfies (7). This checking is relatively simple to perform for POs, since the infinite sum in (6) reduces to a finite one (of length p). In the crystalline cases, the POs of M^n ($n=3,4,6$) can be defined in the toroidal phase space. When "lifted" to the full phase space, these POs satisfy, in general, the relation $v_{s+p}=v_s+e_s$, $s=0,\ldots,p-1$, where p is a multiple of n and e_s are integers [21]. The POs with $e_s\neq0$ are "accelerator modes." These crystalline cases, both in the toroidal and the full phase space frameworks, can be studied using the natural code $b_s=c_s$ [22], as in the case of TC maps [6,7].

It is easy to show that sequences having the property that $|b_s+(2-\eta)x_c|\leq|\eta+\epsilon|/2-1$ satisfy (7) and are therefore admissible. It follows then from (8) and (10) that all the sequences **c** with c_s chosen arbitrarily in the interval

$$1-|\eta+\epsilon|/2<c_s+(2-\eta)x_c<|\eta+\epsilon|/2-1-\beta \quad (12)$$

are admissible and form a proper subspace $\Sigma_{R,c}$ of Σ_c for all (a_{-1},a_0). The set of orbits corresponding to $\Sigma_{R,c}$ is a full horseshoe with zero measure and finite fractal dimensionality [22]. Such "random" chaotic sets usually exhibit a rigorously diffusive behavior [11]. We define the diffusion coefficient as

$$D=\alpha^2\lim_{s\to\infty}\frac{\langle(u_s-u_0)^2+(v_s-v_0)^2\rangle_{\mathcal{E}}}{2s}, \quad (13)$$

where $\langle\ \rangle_{\mathcal{E}}$ denotes average over an ensemble $\mathcal{E}$ of initial conditions (u_0,v_0). Clearly, D is proportional to the average growth rate of the kinetic energy in (1). It is interesting to study the dependence of D on α, which is proportional to the magnetic field B. The function $D(\alpha)$ will be calculated, naturally, at fixed chaos strength, i.e., at fixed λ. In order that $D(\alpha)$ will characterize a genuine diffusive behavior, the ensemble $\mathcal{E}$ will be chosen as the random chaotic set Λ corresponding to the space $\bar{\Sigma}_{R,c}$ of sequences satisfying (12) with $\beta=1$. This space is simply the intersection of the spaces $\Sigma_{R,c}$ for all α. The topological entropy of Λ is $h=\ln N$, where N is the number of values assumed by c_s in (12) for $\beta=1$. In the crystalline cases, we find the exact result [22]: $D=\alpha^2\xi(N^2-1)$, where $\xi=1/24$ for $n=4$ and $\xi=1/18$ for $n=3,6$. In all other cases, D is calculated accurately (without roundoff errors, to "stay" in Λ) by producing first a large ensemble of long sequences **c** in $\bar{\Sigma}_{R,c}$ using a random-number generator. The corresponding values of (u_s,v_s) in (13) are then calculated from Eqs. (9), (10), and (6), with $a_{-1}=a_0=0$. The estimated error in this calculation is not larger than 1%. The results for two sets Λ are presented in Fig. 2, together with standard results obtained by direct iteration of the map (2) using a "realistic" uniformly distributed ensemble $\mathcal{E}$. While the latter results contain, of course, large roundoff errors, they may be reliable due to the shadowing effect in a uniformly hyperbolic system. The discrepancy between these results and the corresponding ones for $\mathcal{E}=\Lambda$ should reflect the effect of all the admissible sequences which are not included in the random chaotic sets. We see that this discrepancy and $D(\alpha)$ increase without limit as α approaches the value of π, for which (1) is integrable and exhibits ballistic motion. On the other hand, as α approaches 0, these quantities decrease and tend rapidly to well-defined limits. The limit values are precisely those corresponding to the TC sawtooth map, since the $\alpha\to0$ limit of the web map is a TC map [15]. In any case, as $|\lambda|$ is increased, the accurate results for $D(\alpha)$ obtained using the random ensembles $\mathcal{E}=\Lambda$ should approach those obtained using more "realistic" (uniform) ensembles. The latter ensembles may be approximated by sets of periodic orbits of sufficiently long period [9,11], and the corresponding values of $D(\alpha)$ can be calculated using the symbolic dynamics, as

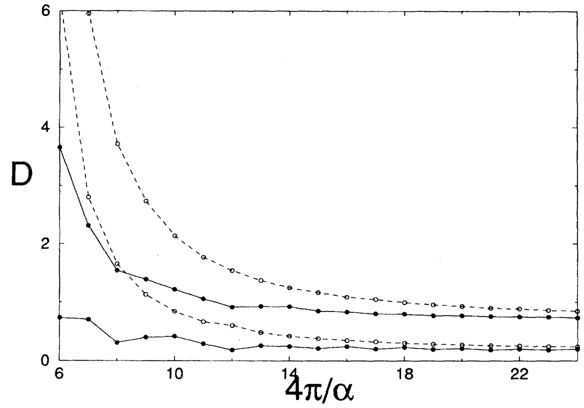

FIG. 2. $D(\alpha)$ for $4\pi/\alpha=6,\ldots,24$ and $x_c=0$. The solid lines are results obtained from the symbolic dynamics using chaotic sets corresponding to 60 000 random sequences **c** of length 6000 with $c_s=-1,0$ (lower line) and $c_s=-2,-1,0,1$ (upper line). These sequences are admissible for $\lambda\geq3.73$ and $\lambda\geq5.83$, respectively [see (12)]. The dashed lines are results obtained using a 200×200 uniformly distributed ensemble and 600 iterations of the map (2) for $\lambda=3.73$ (lower line) and $\lambda=5.83$ (upper line).

in the case of the TC sawtooth map [9]. The results of these calculations will be presented elsewhere [22].

In conclusion, we have introduced an exact symbolic dynamics for a model of strong chaos on stochastic webs with arbitrary quasisymmetry. This symbolic dynamics turns out to have unique features, necessary for describing an extended Hamiltonian system without translational invariance in phase space. We have shown how the infinite alphabet for the basic code can be effectively reduced to a finite one. The symbolic dynamics becomes then useful to study unexplored aspects of the rich dynamical problem of charged particles interacting with an electrostatic wave packet in a magnetic field. In this work, the global diffusion rate of random chaotic sets was calculated accurately as a function of the resonance parameter, and other aspects of the problem will be studied elsewhere [22].

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[21] One has [22] $e_s=\Sigma_{j=0}^{k\mu-1}(-1)^{\mu+(\mu-1)j}(b_{jn/\mu+s+1}-b_{jn/\mu+s+2})$, where $k=p/n$ (integer), $\mu=1$ for $n=3,4$, and $\mu=2$ for $n=6$.

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